Calculus, *Concept of limit*

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The most important, fundamental concept of calculus is limit. The base of limit is the con-tinuity of the real number set $\mathbb{R}$, which would be skipped here. Coming from limit, we can get differential, integral, series and so on. If we must compare which part of calculus is most im-portant, limit. Then comes differential, Riemann integral may be the least important, while Lebesgue measure theory is much better.

For I may have no other words to write during the military training, I am going to write something about calculus. Today should be the first day.

The basic description of limit is for sequence, and then expand to functions. If we take a board view, we can regard differential, Riemann integral as two different special limits. Thus limit is very important, and the brilliant methods of describing a limit is also significant.

There are two statements to describe a **sequence** limit:

$\epsilon - N$ definition

$\lim_{n\to\infty} x_n = x_l \Leftrightarrow$

$$\forall \epsilon \in \mathbb{R}^+, \exists N_\epsilon \in \mathbb{N}, |x_n - x_l| < \epsilon$$

Cauchy's Convergence Test

$\exists \lim_{n\to\infty} x_n \in \mathbb{R} \Leftarrow$

$$\forall \epsilon \in \mathbb{R}^+, \forall n, m > N_\epsilon \in \mathbb{N}, |x_n - x_l| < \epsilon$$

These two statements offer a profound observation in how continuous $\mathbb{R}$ is and how could that continuity lead to limit.

$\epsilon - N$ definition tells us that for any positive number ϵ, the *distance* between the sequence x_n and some number $x_l \in \mathbb{R}$ can be smaller than it, which means they can get close to each other to any extent. We can use such phenomenon or behavior to describe a limit. Such properties obviously depend on the continuity of real number set (proved by *Dedekind's Theorem*). If not give the continuity (supposing you have known how it is defined), we will have no confidence to say that

there can always exist a number between ϵ and 0. Even if we get the continuity of $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$, we cannot assure it.

Cauchy's Convergence Test, however, describes such phenomenon or behavior in another way, not needing the fixed number x_l. It requires the difference in itself be convergent to zero as $n \to \infty$. *Cauchy's Convergence Test* reveals the properties of $\mathbb{R}$ better than $\epsilon - N$ *definition*, and can be proved through the concept of *Infimum* and *supremum* (continuity), and *Bolzano-Weierstrass Theorem*.

Situations are similar in **functions**, but there are three statements and the equality among them can be proved based on continuity and the properties of continuous functions on a closed interval.

$\epsilon - \delta$ *definition*

$\lim_{x\to x_0} f(x) = L \Leftrightarrow$

$$\forall\epsilon \in \mathbb{R}^+, \exists\delta_\epsilon \in \mathbb{R}^+, \forall x \in \dot{B}_{\delta_\epsilon}(x_0) = \{x|0 < |x - x_0| < \delta_\epsilon\}, |f(x) - L| < \epsilon$$

Heine's Theorem

$\exists \lim_{x\to x_0} f(x) = L \Leftrightarrow$

$$\forall\{x_n\} : \lim_{n\to\infty} x_n = x_0, x_n \neq x_0, \exists \lim_{n\to\infty} f(x_n) = L$$

Cauchy's Convergence Test

$\exists \lim_{x\to x_0} f(x) \in \mathbb{R} \Leftarrow$

$$\forall\epsilon \in \mathbb{R}^+, \exists\delta_\epsilon \in \mathbb{R}^+, \forall x_1, x_2 \in \dot{B}_{\delta_\epsilon}(x_0), |f(x_1) - f(x_2)| < \epsilon$$

The statements for function limit are very similar to the sequence ones. So the inner theory about continuity does not require repeat. And in the function situation, we have a more adapta-ble calculating method: *Taylor's Theorem* and *L'Hopital's Rule* (actually it should be called *Bernoulli's Rule*). So *Cauchy's Convergence Test* is not providing a definite limit number (which also means a more extensive adaptability) and $\epsilon - \delta$ *definition*'s being difficult to calculate does not matter.

All the situation are similar when it is expanded to a multi-dimension Euclidean Space. The only thing need to be paid attention is about the definition of distance of two vectors (expressed in bold font)

$$\varphi(\mathbf{x_1}, \mathbf{x_2}) : \mathbb{R}^n \to \mathbb{R}, \quad \theta(\mathbf{x_1}, \mathbf{x_2}) : \mathbb{R}^m \to \mathbb{R}$$

$\epsilon - \delta$ *definition*

$\lim_{\mathbf{x}\to\mathbf{x_0}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in \mathbb{R}^m, \mathbf{x}, \mathbf{x_0} \in \mathbb{R}^n \Leftrightarrow$

$$\forall\epsilon \in \mathbb{R}^+, \exists\delta_\epsilon \in \mathbb{R}^+, \forall\mathbf{x} \in \dot{B}_{\delta_\epsilon}(\mathbf{x_0}) = \{\mathbf{x}|\varphi(\mathbf{x}, \mathbf{x_0}) < \delta_\epsilon\}, \theta(\mathbf{f}(\mathbf{x}) - \mathbf{L}) < \epsilon$$

Cauchy's Convergence Test

$$\exists \lim_{\mathbf{x}\to\mathbf{x_0}} \mathbf{f}(\mathbf{x}) \in \mathbb{R}^m, \mathbf{x} \in \mathbf{R}^n \Leftarrow$$

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta_\epsilon \in \mathbb{R}^+, \forall \varphi(\mathbf{x_1}, \mathbf{x_2}) < \delta_\epsilon, \theta(\mathbf{f}(\mathbf{x_1}), \mathbf{f}(\mathbf{x_2})) < \epsilon$$

Distance mapping $\varphi(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$ can be any kind of mapping from high-dimensional space to real number field as long as they meet the basic requirements for *distance*.