Calculus, *Infinitesimal*

2015-07-12

Infinitesimal is used to state or describe a number that is too small to measure or can be small to any wanted extent. The word came from 17th century Latin. The plain understanding of infinitesimal can date back to Archimedes'time, but the modern conception was introduced by Leibniz in 1670s.

To make a simple description of infinitesimal α

$$\forall \epsilon \in \mathbb{R}^+, 0 < |\alpha| < \epsilon$$

This description tells that for any given positive number ϵ, $0 < |\alpha| < \epsilon$. So infinitesimal can be close to zero to any extent, and so little that it is smaller than any positive number. From this description, we can see that infinitesimal is not a definite real number but a tendency, a behav-ior about a variable. So it is not appropriate to use the description above as a definition, for it does not contain the change. Infinitesimal is formally defined by the sequence limit:

$$\lim_{n \to \infty} x_n = 0$$

Then x_n is an infinitesimal as $n \to \infty$.

It is not difficult to realize that such infinitesimal contains the most significant knowledge about limit. For a normal sequence limit, we can express that the difference between the sequence and its limit is an infinitesimal variable as $n \to \infty$. For a function limit, we can also express that the difference between function value and its limit is an infinitesimal variable as $x \to x_0$

$$\alpha = x_n - x_l, \beta = f(x) - L$$

Attention should be paid that infinitesimal variable is defined by limit, so we cannot use infinitesimal back to define limit, otherwise we would be lost in a vicious circle. But it is still very useful especially in differential, *Taylor's Theorem* and to calculate limit through the theorem.

As I said before, infinitesimal is a core part of limit, so it is very natural to use infinitesimal to describe continuity. In logical order, we should prove the continuity of $\mathbb{R}$ first and then give the concept of limit and infinitesimal, but through the statement of continuity's base, Infimum and supremum, we can understand how infinitesimal variable can exist.

For any bounded and nonempty set of real numbers $\mathbb{S}$, we define:

$$infimum : \inf \mathbb{S} = \max\{m | \forall s \in \mathbb{S}, m \leq s\}$$

$$supremum : \sup \mathbb{S} = \min\{M | \forall s \in \mathbb{S}, s \leq M\}$$

Because $\mathbb{R}$ is continuous, so $\inf$ and $\sup$ exist. Then we have:

$$\forall \epsilon \in \mathbb{R}^+, \exists \hat{s} \in \mathbb{S} : \sup \mathbb{S} - \epsilon < \hat{s}$$

$$\forall \epsilon \in \mathbb{R}^+, \exists \tilde{s} \in \mathbb{S} : \tilde{s} < \sup \mathbb{S} + \epsilon$$

The expressions above omit the proof of the continuity and properties. But we can generally see infinitesimal from the last two properties. The difference between a definite number, $\sup \mathbb{S}$, and **some** elements $\tilde{s}$ in the set can be as close as possible.

Continuity of $\mathbb{R}$ can be proved by the .existence of infimum and supremum. And in this concept, we see the infinitesimal $\sup \mathbb{S} - \tilde{s}$. But attentions must be paid to set element $\tilde{s}$, for not every $s \in \mathbb{S}$ can be the appropriate $\tilde{s}$. The selection, or constraint on the elements, should be the process of limit. In this way, continuity of $\mathbb{R}$, infinitesimal, limit are combined altogether.

EXAMPLE

Bounded and nonempty set of real numbers $\mathbb{S} = \left\{\frac{1}{n}\right\}, n = 1, 2, \ldots$

Without proving, $\inf \mathbb{S} = 0$. Thus,

$\forall \epsilon \in \mathbb{R}^+, \exists \tilde{s} \in \mathbb{S}, s.t.$

$$\tilde{s} = \frac{1}{n} < \inf \mathbb{S} + \epsilon = 0 + \epsilon$$

$$\therefore \tilde{s} \in \tilde{\mathbb{S}} = \left\{\tilde{s} \mid \tilde{s} = \frac{1}{n}, n > N_\epsilon = \left[\frac{1}{\epsilon}\right]\right\}, \quad \tilde{\mathbb{S}} \subset \mathbb{S}$$

Infinitesimal $\alpha = |\inf \mathbb{S} - \tilde{s}| = \frac{1}{n}$

For infinitesimal variables are tendencies, behaviors while not numbers, we cannot figure them out but only compare them. Compare which infinitesimal goes to zero faster than another and rank the order of infinitesimal. This part of statement is more a differential thing (we can define derivative as the rate of two infinitesimal variables) so it is not covered in limit.

Now we see it, the expression $\forall \epsilon \in \mathbb{R}^+, 0 < |\alpha| < \epsilon$ is a little odd and not scientific. If we look deep into continuity, we can find the natural expression of infinitesimal. And if we look deeper, we can find the limit process, how it is closely associated with infinitesimal and recognize properly that infinitesimal is a kind of variable.