Calculus, *Uniformly Continuous Function*

2015-07-13

It's easy to understand how a function can be continuous at a point:

Set $\mathbb{X} = \{x|\mathbb{X} \text{ takes } x_0 \text{ as limit point }\}$. $x \in \mathbb{X}$,

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Or defined through increment or infinitesimal:

$$\lim_{\mathbf{\Delta x}\to\mathbf{0}} \mathbf{f}(\mathbf{x_0}+\mathbf{\Delta x}) - \mathbf{f}(\mathbf{x_0}) = \mathbf{0}$$

The font are bold here because we take it as a general multi-variable mapping, which in-cludes the single-variable function situation. Actually, whether we can and how to figure out this infinitesimal is an issue of differential.

But we are talking about the continuity at a point. This makes the property seems a little weak. To get a stronger condition in an interval, we define *Uniform Continuity*:

Multi-variable mapping $\mathrm{f}(\mathrm{x})$ defined in an interval $\mathbb{X} \subset \mathbb{R}^n$.

$\forall\epsilon \in \mathbb{R}^+, \exists\delta_\epsilon \in \mathbb{R}^+, \forall\tilde{\mathrm{x}}, \hat{\mathrm{x}} \quad s.t. 0 < |\tilde{\mathrm{x}} - \hat{\mathrm{x}}|_{\mathbb{R}^m} < \delta_\epsilon$,

$$|\mathrm{f}(\tilde{\mathrm{x}}) - \mathrm{f}(\hat{\mathrm{x}})|_{\mathbb{R}^n} < \epsilon$$

From the definition, we see if a function $\mathrm{f}(\mathrm{x})$ is uniformly continuous, $\mathrm{f}(\tilde{\mathrm{x}})$ and $\mathrm{f}(\hat{\mathrm{x}})$ can be close to each other to any extent ϵ, as long as we assure that $\tilde{\mathrm{x}}$ and $\hat{\mathrm{x}}$ are sufficiently close to each other by δ_ϵ only.

Thanks to Bolzano, Weierstrass, and their theorems, which are omitted here, we can get this very important and famous theorem:

Cantor's Theorem

If multi-variable mapping $\mathrm{f}(\mathrm{x})$ *is continuous in a bounded closed set* $\mathbb{X} \subset \mathbb{R}^n$*, then* $\mathrm{f}(\mathrm{x})$ *is uniformly continuous in* $\mathbb{X}$*.*

Uniform continuity is a kind of very significant property of function and is a sufficient condition for many later analysis.