Calculus, *Differential and Derivative*

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First of all, we must introduce the brilliant notation for infinitesimal from *Landau*. Landau's symbol, $o(x)$, is defined as below:

Supposing x is an infinitesimal, then $o(x)$ is also an infinitesimal but whose order is higher, which means $o(x)$ goes faster than x to zero. This is a comparison of infinitesimal.

$$\lim_{x\to 0}\frac{o(x)}{x}=0$$

With the help of Landau's symbol, we can return to yesterday's expression of continuous function and rewrite it as,

$$\lim_{\Delta x\to 0} f(x_0+\Delta x)-f(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{1}=0$$

If we use Landau's symbol to express it:

$$f(x_0+\Delta x)-f(x_0)=o(1)$$

Take a deeper look into the expression, we can relate $o(1)$ with Δx as $o(\Delta x^0)$. So a function being continuous at a point can be expressed through infinitesimal:

$$f(x_0+\Delta x)-f(x_0)=o(\Delta x^0)$$

However, such way of studying is very rough, too rough to study the properties of functions, in fact. So we hope it show more details about this infinitesimal

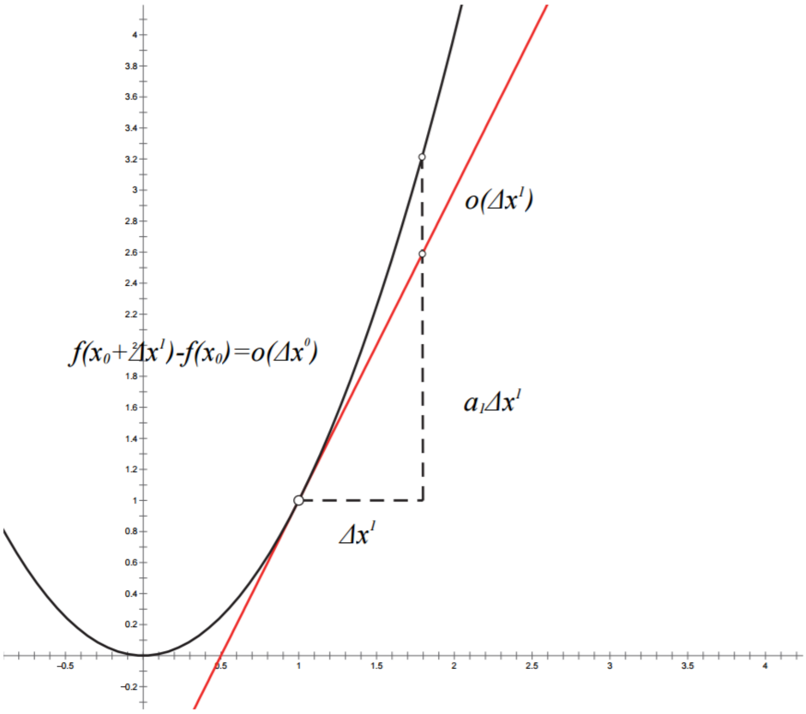
$$f(x_0+\Delta x)-f(x_0)=o(\Delta x^0)=a_1\cdot\Delta x^1+o(\Delta x^1)$$

If the infinitesimal is expanded to such form, we would have a major linear part and a higher-order infinitesimal $o(\Delta x^1)$

Luckily, such way of expansion has very distinct geometry meaning: **a line**,

$$f(x)=f(x_0)+a_1(x-x_0)+o(\Delta x^1),\Delta x^1=x-x_0$$

As any textbook, I use the illustration below to introduce differential, derivative and their meanings.



This illustration gives a geometrical explanation of the relationship between major linear part and higher-order infinitesimal variable. Then the question facing us is how to get the coefficient a_1. The method is called ***derivative***[1]. Its principle is not difficult:

$$\lim_{\Delta x^1 \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x^1} = \lim_{\Delta x^1 \to 0} \frac{a_1 \Delta x^1 + o(\Delta x^1)}{\Delta x^1} = a_1$$

$$\frac{dy}{dx} = f^{(1)}(x_0) = a_1$$

If we look into this process (made independently through the analysis of infinitesimal), we can find it actually turns the secant line into a tangent line through limit method (geometry). So though a_1 may take different values, the only value allowing $f(x_0 + \Delta x) - f(x_0) - a_1 \Delta x^1$ always a higher-order infinitesimal than Δx^1 is derivative, and the line should be tangent.

While, often be mixed, the major linear part $f(x_0 + \Delta x) - f(x_0) - o(\Delta x^1) = f^{(1)}(x_0)\Delta x^1$ is the differential of function. Though the higher-order infinitesimal variable $o(\Delta x^1)$ is very small, it cannot be omitted unless we do derivative. We notate it as:

$$\lim_{\Delta x^1 \to 0} f(x_0 + \Delta x) - f(x_0) - o(\Delta x^1) = \lim_{\Delta x^1 \to 0} f^{(1)}(x_0)\Delta x^1$$

[1] As I said before, this method is a kind of comparison between the infinitesimal variables(between $f(x_0 + \Delta x) - f(x_0)$ and Δx^1). If we have two infitesimal variables, we compare them to judge their relative order: $p = \lim \frac{\alpha}{\beta}$, as $\alpha \to 0, \beta \to 0$.

$$\Rightarrow \lim_{\Delta x^1 \to 0} \Delta y - o(\Delta x^1) = \lim_{\Delta x^1 \to 0} f^{(1)}(x_0)\Delta x^1 \Rightarrow dy = f^{(1)}(x_0)dx$$

So in single-variable situation, the method to test differential is:

$$\lim_{\Delta x^1 \to 0} \frac{f(x_0 + \Delta x) - f(x_0) - f^{(1)}(x_0)\Delta x^1}{\Delta x^1} = \lim_{\Delta x^1 \to 0} \frac{o(\Delta x^1)}{\Delta x^1} = 0$$

This test is important because for multi-variable situation, we still use this relationship to test differential.

$$\lim_{\mathbf{\Delta x} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{x_0} + \mathbf{\Delta x}) - \mathbf{f}(\mathbf{x_0}) - D\mathbf{f}(\mathbf{x_0})(\mathbf{\Delta x})}{|\mathbf{\Delta x^1}|_{\mathbb{R}^m}} = \lim_{\mathbf{\Delta x} \to \mathbf{0}} \frac{\mathbf{o}(|\mathbf{\Delta x}|_{\mathbb{R}^m})}{|\mathbf{\Delta x}|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^n}$$

$D\mathbf{f}(\mathbf{x_0})(\mathbf{\Delta x})$ is ***Jacobian Matrix*** of multi-variable mapping $\mathbf{f}(\cdot)$. The elements of Jacobian Matrix are *Partial Derivative*, which can be got by fixing all other variables thus allowing only one variable to change.

From the discussion above, we have showed more detail in the very rough infinitesimal $o(\Delta x^0)$ in continuous function, as long as derivative is available at the point. If we need more detail, the function would be required to have derivative till p order.

$$f(x_0 + \Delta x) - f(x_0) = \sum_{k=1}^{p} \frac{f^{(k)}(x_0)}{k!}\Delta x^k + o(\Delta x^p)$$

This is normally called ***Taylor's Theorem***. I may cover its proof in later days. This theo-rem is usually regarded as a method of using a polynomial to approach a very good function in the neighborhood of a given point. But here, I regard continuous function, differential and derivative, and Taylor polynomial as details of $f(x_0 + \Delta x) - f(x_0)$ to different extent. Cotinuity is rough, differential is better, Taylor polynomial is the most detailed.