Calculus, *Multi-variable Differential*

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Multi-variable differential and single-variable situation have great similarity. Such simi-larity mainly relies on the similarity of limit process, while the similarity of limit process mainly relies on the simple inequality below, which is actually the reflection of the structure of Euclidean Space:

$$\forall i = 1, 2, \cdots, m; p = 2 \quad |a_i| \leq \left(\sum_{j=1}^{m} |a_j|^p \right)^{\frac{1}{p}} \tag{1}$$

This very basic relationship is the great bridge between multi-variable limit and single-variable limit. So we can make a deeper analysis about what I said on 2015-07-11:

$\forall i = 1, 2, \cdots, m; p = 2$

$$\varphi(\tilde{x}_i, \hat{x}_i)_{\mathbb{R}} = |\tilde{x}_i - \hat{x}_i| \leq \left(\sum_{j=1}^{m} |\tilde{x}_i - \hat{x}_i|^p \right)^{\frac{1}{p}} = \varphi(\tilde{\mathbf{x}}, \hat{\mathbf{x}}_{\mathbf{i}})_{\mathbb{R}^m} < \epsilon \tag{2}$$

From (2), we can easily understand if we give a constraint $\varphi(\cdot, \cdot)_{\mathbb{R}^m} \in \mathbb{R}$ for $\mathbb{R}^m$ space, according to the inequality (1), we can have the same constraint on the component $\varphi(\cdot, \cdot)_{\mathbb{R}^m}$. Such property guarantees the components on every axis (projection) can reach a common infinitesimal constraint ϵ. So limits on every component can be got.

And if we turn back the order,

$\forall i = 1, 2, \cdots, m; p = 2$

$$\varphi(\tilde{x}_i, \hat{x}_i)_{\mathbb{R}} = |\tilde{x}_i - \hat{x}_i| < \epsilon \Rightarrow \varphi(\tilde{\mathbf{x}}, \hat{\mathbf{x}})_{\mathbb{R}^m} = \left(\sum_{j=1}^{m} |\tilde{x}_i - \hat{x}_i|^p \right)^{\frac{1}{p}} < \sqrt[p]{m} \cdot \epsilon$$

The normal distance can also be controlled. Such relationships mainly rely on the property of absolute value, or the property of being non-negative, and such is the property of **Normal Linear Space**[1].

So, it is lucky that we expand the conclusions of single-variable differential in Euclidean Space (a kind of Metric Space), so we enjoy such consistence.

However, even the two situations share great similarity, there is tiny but significant difference between them. One of them is the relationship between differential and derivative: in single-variable situation, the existence of differential and derivative is equal. But in multi-variable, we cannot get the former from the latter.

To simplify our discussion, we constrain the normal mapping $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ to a function $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}$. Then, for any variable x_k, we can fix other variables $\{x_j\}_{j=1,2,\ldots,m;j\neq k}$ and have its partial derivative:

$$\frac{\partial f}{\partial x_k}(x) = \lim_{\Delta x_k \to 0} \frac{f(x_1, \ldots, x_k + \Delta x_k, \ldots, x_m) - f(x_1, \ldots, x_k, \ldots, x_m)}{\Delta x_k}$$

If we take the fixed variables as constants, then the changes are constrained to one axis, this partial derivative is a normal derivative for function $g(x_k) = f(x_1, \ldots, x_k, \ldots, x_m)$. Such is the ordinary situation of single-variable differential. If we assure the existence of partial derivative of every variable, written in the increment relationship below,

$\forall i = 1, 2, \cdots, m$

$$f(\tilde{\mathbf{x}} + \Delta x_k) - f(\tilde{\mathbf{x}}) = \frac{\partial f}{\partial x_k}(\tilde{\mathbf{x}})\Delta x_k + o(\Delta x_k)$$

Can we have the existence of the **Exact Differential**? The answer is No.

If we separate the exact difference into many increment and adapt partial derivative

$$f\left(\tilde{\mathbf{x}} + \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_m \end{bmatrix}\right) - f(\tilde{\mathbf{x}}) = \sum_{j=1}^{m} \left(f\left(\tilde{\mathbf{x}} + \begin{bmatrix} * \\ \Delta x_j \\ \vdots \end{bmatrix}\right) - f\left(\tilde{\mathbf{x}} + \begin{bmatrix} * \\ \Delta x_{j+1} \\ \vdots \end{bmatrix}\right)\right)$$

$$= \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} \left(\tilde{\mathbf{x}} + \begin{bmatrix} * \\ \Delta x_{j+1} \\ \vdots \end{bmatrix}\right) \Delta x_j + \sum_{j=1}^{m} o(\Delta x_j)$$

$$\text{make } \begin{bmatrix} * \\ \Delta x_{j+1} \\ \vdots \end{bmatrix}_{j=m} = \mathbf{0}$$

[1]When this article is written, I consulted the book of *Functional Analysis*, and found this is a very basic conclusion in Metric Space.

For the partial derivative part, we see unless the partial derivative part is continuous, we wont have the existence of exact differential. On the other hand, the infinitesimal reminder can be controlled through what we have talked about the similarity between the two situations. Of course the standard method is to use ***Lagrange Mean Value Theorem*** and so need not to analysis the infinitesimal reminder.

This conclusion reveals what is hidden in single-variable situation. If all the stories happen on only one axis, we cannot see the truth. When it comes to a high-dimension situation, we find that it seems that partial derivative cannot converge on one point.