Calculus, *Linear Algebra and Jacobian Matrix*

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Like Prof. Xie said this semester, multi-variable differential must be closely related to linear algebra. As far as I have learned, the calculus is something happened in Euclidean Space, which is a part of linear space $\mathbb{L}$ satisfying 8 properties, you can consult these requirements on any algebra book.

In fact, if we take a deeper view into these seemed normal properties, we can find some-thing very interesting: requirement $\forall \alpha, \beta \in \mathbb{L}, \alpha + \beta = \beta + \alpha$ reveals that all the elements in linear space have their equal positions. So, in linear space, every direction has the same property. And many other conclusions which wont be discussed here. Euclidean Space is a special kind of space where inner product is defined.

For multi-variable mapping $\mathbb{R}^m \to \mathbb{R}^n$, we separate them into n multi-variable functions. Derivative each function so we have the exact differential.

$$d\mathbf{f}(\tilde{\mathbf{x}}) = \begin{bmatrix} df_1(\tilde{\mathbf{x}}) \\ \vdots \\ df_n(\tilde{\mathbf{x}}) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \tilde{x}_1} & \cdots & \frac{\partial f_1}{\partial \tilde{x}_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial \tilde{x}_1} & \cdots & \frac{\partial f_n}{\partial \tilde{x}_m} \end{bmatrix} \cdot \begin{bmatrix} d\tilde{x}_1 \\ \vdots \\ d\tilde{x}_m \end{bmatrix}$$

$$\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_m)} \cdot d\tilde{\mathbf{x}} = D\mathbf{f}(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}}$$

In this differential relationship, we call the coefficient matrix $D\mathbf{f}(\tilde{\mathbf{x}}) \in \mathbb{R}^{n \times m}$ as **Jacobian Matrix** to memorize the great mathematician *C.G.J.Jacobi*, whose research showed that the matrix has the same properties as a normal single-variable derivative or partial derivative. Under such matrix notation, the multiplication of matrix is especially similar to the theorem of function composition derivative.

We find that the row vectors is actually Hamiltonian (∇), or gradient (Hamilton with number multiplication): $D\mathbf{f}(\tilde{\mathbf{x}}) = [\nabla f_j(\tilde{x})]^T_{j=1,2,\ldots,n}$. And the column vector is tangent vector of an n-dimension curve Γ: $\mathbb{R} \ni x_j \to [f_j(\tilde{x}_l)]^T_{j=1,2,\ldots,n} \in \mathbb{R}^n$, $D\mathbf{f}(\tilde{\mathbf{x}}) = [\tau(\tilde{x}_l)]^T_{j=1,2,\ldots,m}$

Now we study the differential relationship: $d\mathrm{f}(\tilde{\mathrm{x}}) = D\mathrm{f}(\tilde{\mathrm{x}}) \cdot d\tilde{\mathrm{x}}$. If we put this relationship in the linear algebra situation, we can have many interesting thoughts.

First, this relationship can be taken as a typical linear mapping between two different linear spaces. Then we can study it as a linear mapping.

Second, considering that elements in vectors on both sides are all infinitesimal variables, we may take this relationship as homogeneous linear equations and so $d\tilde{\mathrm{x}}$ is the solution space. And not only that, since homogeneous linear equation is only a rough description (may be wrong), we may estimate its difference to the real situation.

Third, look back on Calculus 04, then we may expand to second-order and get **Hesse Matrix**. If this matrix is taken as a quadratic form, it would be very useful to judge the extremum (thanks to Taylor's Theorem).

Fourth, when we have the mapping $\mathrm{f} : \mathbb{R}^m \to \mathbb{R}^n$, we say we have the transformation. Then Jacobian Matrix should be a square matrix, the linear mapping should be linear transformation. Then we can have its **eigenvalue** (but I dont know its usage now). This part should contain the most content.

I may gradually try to solve these problems or guesses. Since I know my interest is in space, I will not write more about Taylor's Theorem unless I find it is related to this theme.