Calculus, *Linear Equations and Jacobian Matrix*

2015-07-18

From today, I will try to solve some of the questions proposed yesterday. But I dont know whether I would have some conclusions or not. Yet the LINEAR ALGEBRA I learnt in 1st semester is far from enough and I almost forget them, so I have to review and explore some algebra knowledge. Anyway, I will try my best in spare time.

If we take $D\mathbf{f}(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}} = d\mathbf{f}(\tilde{\mathbf{x}})$ as non-homogeneous linear equations or normal matrix equation, we can study this equation through the knowledge in linear algebra. With that knowledge, the necessary and sufficient condition of this equation has (only one) solution is:

$$r = Rank\{D\mathbf{f}(\tilde{\mathbf{x}})\} = Rank\left\{\begin{bmatrix} D\mathbf{f}(\tilde{\mathbf{x}}) & d\mathbf{f}(\tilde{\mathbf{x}}) \end{bmatrix}\right\} (= m)$$

Of course, this is not a general linear equation question. In the situation of derivative relationship, if the multi-variable mapping has good property, there must be only one solution at a specific point.

Also, such condition tells us that Jacobian Matrix should be a full column rank matrix. So tangent vectors of the curve (column vectors of Jacobian Matrix) $\left[\tau(x_j)\right]_{j=1,2,\ldots,m}$ should be **linear independent**. This is reasonable since these tangent vectors take different directions. And if the dimension meets the special condition m=n, these tangent vectors can be the basis of codomain, or image space.

Particularly, we now study four special linear mapping:

function: $\mathbb{R}^n \supset \mathbb{D}_x \ni \mathbf{x} \to f(\mathbf{x}) \in \mathbb{R}$;

curve: $\mathbb{R} \supset \mathbb{D}_x \ni x \to \mathbf{f}(x) \in \mathbb{R}^n$;

transition mapping: $\mathbb{R}^k \supset \mathbb{D}_x \ni \mathbf{x} \to \mathbf{f}(\mathbf{x}) \in \mathbb{R}^n, \quad k \in (1, n-1)$;

surface: $\mathbb{R}^{n-1} \supset \mathbb{D}_x \ni \mathbf{x} \to \mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$;

and transformation: $\mathbb{R}^n \supset \mathbb{D}_x \ni \mathbf{x} \to \mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$

The first three mappings belong to **Riemann Manifold**, while the last is **Euclidean Manifold**. The last one is complicated and I will cover it tomorrow.

It is not difficult to talk about the function mapping. Its Jacobian Matrix is just a gradient row vector $\nabla f(\tilde{\mathbf{x}})$. Then $df(\tilde{\mathbf{x}}) = Df(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}} = \nabla f(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}}$ can be taken as a inner product in high dimension Euclidean Space. Now Jacobian Matrix is a vector itself, so it is always full row rank.

For a generalized curve, we can take it as a mapping from a continuous real number interval to a high dimension region. Though this expression is not very strict, we can roughly take $\mathbb{R} \to \mathbb{R}^n$ as a curve mapping. In the curve situation, Jacobian Matrix is a tangent vector of curve. Of course the tangent vector is independent, but only these independent vector is not enough to be the basis of image space. But for such vector matrix, if its rank is not 0, which means its partial derivative elements all exist and some of them have higher derivative, we can confirm that it is **smooth**.

The expansion of the concept of surface is also very natural. Notice that we use 2 variables to describe a surface in $\mathbb{R}^3$, so we generalize the specific mapping from surface $\mathbb{R}^m \to \mathbb{R}^{m+1}$. But notice that though the curve belongs to $\mathbb{R}^{m+1}$, the surface itself is a $\mathbb{R}^m$ description. Then $D\mathbf{f}(\tilde{\mathbf{x}}) \in \mathbb{R}^{m\times(m+1)}$. If the surface is good enough, we would have only one solution to the matrix equation, thus Jacobian Matrix is full column rank. So we can get a $\mathbb{R}^{m\times m}$ identity matrix in Jacobian Matrix, which proves the existence of partial derivative and some of them can have higher derivative so that they are different and wont be eliminated. So the surface is smooth (part meaning of good enough). Vice versa, if Jacobian Matrix is full column rank matrix, the surface is smooth.

EXPERIMENT

When $r = Rand\{D\mathbf{f}(\tilde{\mathbf{x}})\} < m$, we know that the homogeneous linear equation (at least has a zero solution) have non-zero solutions. Now we study a homogeneous linear equation:

$$D\mathbf{f}(\tilde{\mathbf{x}}) \cdot \mathbf{X} = \mathbf{0} \in \mathbb{R}^{1\times n}$$

This is just a thought in my mind, for $d\mathbf{f}(\tilde{\mathbf{x}}) \to \mathbf{0}$, then we may estimate the difference between X and $d\tilde{\mathbf{x}}$. I dont know whether it has any usage. It is just an experiment. While in single-variable situation, this experiment is very simple because the relationship is very linear. But in vector situation, there may be something interesting because it has basic solutions for the system.

According to the theorem of the non-zero solutions of homogenous linear equations, after basic row transformations, some part of Jacobian Matrix will be identity matrix $\mathbf{I} \in \mathbb{R}^{r\times r}$. We will have $m - r$ columns which dont belong to identity matrix. The unknowns in these columns can be set as free variables. To simplify the expression, suppose the last unknowns are free, then the solution is:

$$\left[[\mathbf{X}_i]_{i=1,\dots,r} \mid [\mathbf{X}_j]_{j=r+1,\dots,m}\right]^T = \left[\left[\sum_{k=1}^{m-r} \mu_k^i \cdot \mathbf{F}_k\right]_{i=1,\dots,r} \mid [\mathbf{F}_j]_{j=1,\dots,m-r}\right]^T$$

Now the situation seems out of control: I can set any real number for free unknowns $[\mathbf{F}_i]$, so the basic solution only reflects its structure. But the relationship between free unknowns and constrained unknowns is linear, so if I control the free ones to be $[dx_i]_{i=r+1,\dots,m}$, the constrained are also infinitesimal variables. Thus their difference is still infinitesimal variables (may be in higher order), which means they can be as close as possible.

If we study another nonhomogeneous linear equation estimation $D\mathbf{f}(\tilde{\mathbf{x}}) \cdot \mathbf{X} = \mathbf{E} = [\epsilon]_{i=1,\dots,n} \in \mathbb{R}^{1\times n}$, maybe we can have some more interesting results related to limit.