Calculus, *Linear Transformation and Jacobian Matrix*

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Today I try to get something when Jacobian Matrix is square and the mapping is a trans-formation $\mathbb{R}^m \to \mathbb{R}^m$.

If the transformation $\mathbf{f}(\cdot)$ is good, the matrix equation $D\mathbf{f}(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}} = d\mathbf{f}(\tilde{\mathbf{x}})$ should have solution. In this transformation, we can use rank for solution analysis, but we can also use **Cramer's Rule** and then measure determinant of Jacobian Matrix.

$$det\{D\mathbf{f}(\tilde{\mathbf{x}})\} = det \begin{bmatrix} \frac{\partial f_1}{\partial \tilde{x}_1} & \cdots & \frac{\partial f_1}{\partial \tilde{x}_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \tilde{x}_1} & \cdots & \frac{\partial f_m}{\partial \tilde{x}_m} \end{bmatrix}$$

note as:

$$det\{D\mathbf{f}(\tilde{\mathbf{x}})\} = \frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_m)}$$

If Jacobian Determinant is non-zero, or Jacobian Matrix is non-singular:

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_m)} \neq 0$$

According to Cramer's Rule, there will always be only one solution. And Jacobian Matrix has its inverse matrix $D^{-1}\mathbf{f}(\tilde{\mathbf{x}})$. So we can have $d\tilde{\mathbf{x}} = D^{-1}\mathbf{f}(\tilde{\mathbf{x}}) \cdot d\mathbf{f}(\tilde{\mathbf{x}})$. This equation tells us the existence of inverse mapping and its differential relationship, helps us transfer from image space to variable space.

Also, Jacobian Matrix is full rank, so m column vectors or tangent vectors are independent, which is very natural for different tangent vector takes different direction, and they can be basis of image space for this is a square matrix.

However, if Jacobian Determinant is zero, there would be no solution or countless solu-tions. In all these zero situations, there is a special one when all elements of Jacobian Matrix and image vector are zero. Then the solution space is **null space** and may be critical points for $f_k\tilde{\mathbf{x}}$.

Actually, I read about Prof. Xie's lecture note about *diffeomorphism* today. There are some conclusions I achieved these days. I regret that I didn't read about these materials before writ-ing the article. If I read more, I won't spend much time on exploring the present knowledge. It can be said that articles these days show my ignorance.

So now we can talk a little about $\mathcal{C}^p$ diffeomorphism. There won't be strict definition here, and we are only required to know: (1) spaces, or sets on both sides are open; (2) the mapping is double bijection; (3) the mapping and its inverse are differentiable to p-order, and most important, $\mathbb{R}^m \supset \mathbb{D}_x \ni \mathbf{x} \to \mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$.

So for a $\mathcal{C}^p$ diffeomorphism, its mapping is a transformation and Jacobian Matrix is square. Notice that the mapping is differentiable to p-order, so Jacobian Matrix is definitely non-singular. All the tangent vectors can be a group of basis in $\mathbb{R}^m$. However, such basis have a characteristic which is they are variable according to the differential point. This is **covariance of vectors**, which describes how things change when the basis change, for example, change from basis$[\tau(\tilde{x}_i)]_{i=1,2,\ldots,m}$ to another basis $[\tau(\hat{x}_i)]_{i=1,2,\ldots,m}$. And Jacobian Matrix is a covariant matrix. Only in this situation, we can regard the matrix as a change of basis. The existence of these conclusions is based on the similarity between the differential manifold.

Since Jacobian Matrix is non-singular, we can discuss about its inverse matrix. From the definition of Jacobian Matrix, it is easy to know:

$$D^{-1}\mathbf{f}(\tilde{\mathbf{x}}) = D\mathbf{x}(\mathbf{f}) = [\nabla \mathbf{x}_j(\mathbf{f})]^T_{j=1,2,\ldots,n}$$

Its tangent vectors can also be a group of basis called **contravariant basis**. And according to the definition of inverse matrix, we know that the multiplication of one non-singular matrix and its inverse matrix is an identity matrix, so we have:

$$[\nabla \mathbf{x}_j(\mathbf{f})]^T_{j=1,2,\ldots,n} \cdot [\tau(\tilde{x}_i)]_{i=1,2,\ldots,m} = \mathbf{I} \in \mathbb{R}^{m\times m}$$

If we write about the elements, we will have a dual relationship:

$$< \nabla x_j(\mathbf{f}), \tau(\tilde{x}_i) >_{\mathbb{R}^m} = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1}$$

$< \cdot, \cdot >_{\mathbb{R}^m}$ is the notation of inner product of two vectors in $\mathbb{R}^m$ space.

Another important part of linear transformation is **eigenvalue**. I have no deep understand-ing of eigenvalue, so sadly I cannot write anything. But I googled this topic and saw there is a good use of eigenvalues of Jacobian Matrix to analysis critical points and their stabilities of linear or almost linear system. So it shows the importance of Jacobian Matrix Eigenvalues. But I won't write about it, since it is well studied and may be not related to the linear space.