Calculus, *Addition to Linear Equations and Jacobian Matrix*

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First, we should understand the relationship between column space of a matrix A and system of linear equations $A\mathrm{x} = \mathrm{b}$, and through ¡strong¿projection¡/strong¿ method, we can estimate or describe the best solution to an unsolvable system.

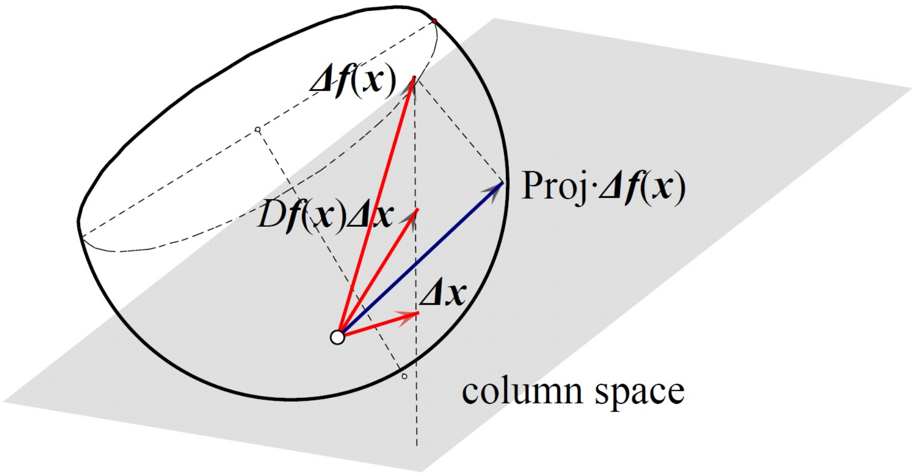
It is not difficult to understand what $A\mathrm{x} = \mathrm{b}$ means:

$$[A.col_1 \cdots A.col_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i(A.col_i) = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

So $A\mathrm{x}$ means a linear combination of columns of matrix A, which is in the ¡strong¿column space¡/strong¿ spanned by the columns. From this point of view, if there exists solution(s) to the system $A\mathrm{x} = \mathrm{b}$, then vector b must be in the column space of A.

Then we return to differential relationship $D\mathbf{f}(\mathrm{x}) \cdot d\mathrm{x} = d\mathbf{f}(\mathrm{x})$. As it is known to all, the columns of Jacobian matrix is tangent vectors at differential point, so the column space of Jacobian matrix is spanned by the tangent vectors, then column space should be ¡strong¿tangent subspace¡/strong¿. In $\mathbb{R}^3$, $z = f(x, y)$, the column space(tangent subspace) should be tangent plane going through the point.

Now we give a more precise description $D\mathbf{f}(\mathrm{x}) \cdot \Delta\mathrm{x}$ instead of $D\mathbf{f}(\mathrm{x}) \cdot d\mathrm{x} = d\mathbf{f}(\mathrm{x})$ to look into the projection estimation. There is no doubt that in most nonlinear cases, $D\mathbf{f}(\mathrm{x}) \cdot \Delta\mathrm{x} \neq \Delta\mathbf{f}(\mathrm{x})$, $\forall \mathrm{x} s.t. \ldots$, which means the increment vector $\Delta\mathbf{f}(\mathrm{x})$ is not in the column space. So we can project vector $\Delta\mathbf{f}(\mathrm{x})$ onto the column space to solve the created system $D\mathbf{f}(\mathrm{x}) \cdot \mathfrak{X} = Proj \cdot \Delta\mathbf{f}(\mathrm{x})$ and estimate the infinitesimal increment.



According to projection theory in linear algebra, the projection matrix $Proj$ onto the column space of Jacobian matrix $D\mathrm{f}(\mathrm{s})$ is:

$$Proj = D\mathbf{f}(\mathbf{x}) \cdot \left(D\mathbf{f}(\mathbf{x})^T \cdot D\mathbf{f}(\mathbf{x}) \right)^{-1} \cdot D\mathbf{f}(\mathbf{x})^T$$

$$\mathfrak{X} = \left(\left(D\mathbf{f}(\mathbf{x})^T \cdot D\mathbf{f}(\mathbf{x}) \right)^{-1} \cdot D\mathbf{f}(\mathbf{x})^T \right) \cdot \Delta\mathbf{f}(\mathbf{x})$$

From such result, we may describe the infinitesimal increment and estimate their difference ¡strong¿without norm or metric¡/strong¿, which may be especially useful in a more general normed linear space when the norm is difficult to figure out. Instead, since we have achieved a linear form through projection, we can use linear combination coefficient.

$$\mathfrak{X} = \left(\left(D\mathbf{f}(\mathbf{x})^T \cdot D\mathbf{f}(\mathbf{x}) \right)^{-1} \cdot D\mathbf{f}(\mathbf{x})^T \right) \cdot \Delta\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \mu_i x_i$$

Maybe we can estimate $\{\mu_i\}_{i=1,\ldots,n}, \mu_i \to 0$ instead of $\|\Delta\mathrm{x}\|_{\mathbb{X}} \to 0$. But as a matter of fact, they should be equivalent since there exists non-degeneracy of norm.

This article is just an addition to the former ones, to take another point of view to look into the differential relationship.